

A geometric Iwatsuka type effect in quantum layers

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Abstract

We study motion of a charged particle confined to Dirichlet layer of a fixed width placed into a homogeneous magnetic field. If the layer is planar and the field is perpendicular to it the spectrum consists of infinitely degenerate eigenvalues. We consider translationally invariant geometric perturbations and derive several sufficient conditions under which a magnetic transport is possible, that is, the spectrum, in its entirety or a part of it, becomes absolutely continuous.

1 Introduction

A homogeneous magnetic field acting on charged particles has a localizing effect, both classically and quantum mechanically. Since numerous physical effects are based on moving electrons between different places, mechanism that can produce transport in the presence of a magnetic field are of great interest. They typically require presence of an infinitely extended perturbation, a standard example being a barrier or a potential wall producing edges states, cf. [7, 8, 13, 14, 15, 16, 17] and references therein. This is not the only possibility, though. In his classical paper [18] Iwatsuka demonstrated that a transport can be induced by a modification of

the magnetic field itself under the assumption of a translational invariance, see also [4].

The aim of the present paper is to show still another mechanism which can produce transport in a homogeneous field for particles confined to a layer with hard walls. As in the case of the Iwatsuka model we will express the effect in spectral terms seeking perturbations that change the Hamiltonian spectrum to absolutely continuous. Our departing point is a flat layer of width $2a$ to which a charged particle is confined and which is exposed to the homogeneous magnetic field perpendicular to the layer plane. The spectrum of such a system is easily found by separation of variables. It combines the Landau levels with the Dirichlet Laplacian eigenvalues in the perpendicular direction, and needless to say that all the resulting eigenvalues are infinitely degenerate, see Sec. 4.2.1 below for more details.

We are going to discuss *geometric perturbations* of such a system, in particular, deformations of the layer which are invariant with respect to translation in a fixed direction. Such layers can be described, e.g., as a set of points satisfying $\text{dist}(x, \Sigma) < a$ where Σ is a surface obtained by shifting a smooth curve which can be parametrized by relation (2.1) below. We are going to derive several conditions which ensure that the unperturbed pure point spectrum will change into an absolutely continuous one. More specifically, our main results can be summarized in the following assertion.

Theorem 1.1. *Let H be the Hamiltonian of a charged quantum particle confined to a layer Ω of a constant width $2a$ in \mathbb{R}^3 built over a C^4 -smooth, translationally invariant surface (2.1) and exposed to a nonzero homogeneous magnetic field pointing in the z -direction. The spectrum of H is purely absolutely continuous if together with technical assumptions $\langle A0 \rangle$ and $\langle A1 \rangle$ any of the following conditions is satisfied:*

- (i) Ω is a one-sided-fold layer, $\lim_{s \rightarrow \pm\infty} x(s) = +\infty$ or $\lim_{s \rightarrow \pm\infty} x(s) = -\infty$. Furthermore, we suppose that the second part of $\langle A3 \rangle$ is fulfilled.
- (ii) Ω is bent and asymptotically flat, $\dot{x}(s) = \alpha_{\pm}$ for all large enough positive and negative s , respectively, where $\alpha_{\pm} \in (0, 1]$. Furthermore, one requires that $\alpha_+ \neq \alpha_-$ and the halfwidth a satisfies the bound described in Lemma 4.4 and Remark 4.5 below.

Moreover, for a fixed $E \in \mathbb{R}$, the spectrum of H below $E + \left(\frac{\pi}{2a}\right)^2$ is absolutely continuous if Ω is thin, i.e. the halfwidth a is sufficiently small, and the generating surface satisfies one of the conditions specified in Proposition 4.8.

The proof of the theorem will be given in Sec. 4, before coming to it we will describe the geometry of the layer and explain the main steps of the argument. Let us add a few remarks. First of all, in Sec. 4.2.1 we demonstrate that the

perturbation must be geometrically nontrivial, because a mere tilt of the layer with respect to the field direction is not enough, with one notable exception. Furthermore, except the claim (i) our condition impose restrictions on the layer thickness. On the other hand, the thinner the layer, the more general deformations we can treat. In particular, the last claim covers perturbations which are compact with respect to the x variable, cf. Proposition 4.8. Note also that the shift by $\left(\frac{\pi}{2a}\right)^2$ in the last claim is needed; without it the claim would be trivial because in a thin layer the spectral threshold is pushed up due to the Dirichlet boundaries.

The method used to treat thin layers is also useful with respect to the original Iwatsuka model and its generalization including addition of a potential perturbation. Recall, in particular, the conjecture stated in [4] according to which *any* nontrivial translationally invariant magnetic perturbation gives rise to the purely absolutely continuous spectrum. Despite a number of sufficient conditions derived after the original Iwatsuka paper [22, 10, 25] to which we add a new one in Theorem 5.1, the question in its generality remains open. In a similar vein, we are convinced that the sufficient conditions we find in this paper are by far not necessary.

2 Preliminaries

2.1 Geometry of the layer

Let Σ be a surface in \mathbb{R}^3 invariant with respect to translation in the y direction and described by means of the following parametrization,

$$\mathcal{L}_0(s, y) = (x(s), y, z(s)) \quad (2.1)$$

with $s, y \in \mathbb{R}$. The functions x and z here are assumed to be smooth enough, unless said otherwise we suppose they are C^4 , and such that $\dot{x}(s)^2 + \dot{z}(s)^2 = 1$, where the dot stands for the derivative with respect to s . The last condition means that the curve $\Gamma : s \mapsto (x(s), z(s))$ in the xz plane is parametrized by its arc length measured from some reference point on the curve. Therefore the (signed) curvature κ of Γ may be fixed as

$$\kappa(s) = \dot{x}(s)\ddot{z}(s) - \ddot{x}(s)\dot{z}(s)$$

and the corresponding unit normal vector to Σ is

$$n(s, y) \equiv n(s) = (-\dot{z}(s), 0, \dot{x}(s)).$$

Let us stress that throughout the paper we assume that

$$\|\kappa\|_\infty < \infty. \quad \langle A0 \rangle$$

If we regard Σ as a Riemannian manifold, then the metric induced by the immersion \mathcal{L}_0 is

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \mu, \nu \in \{s, y\}.$$

Let $a > 0$ and $I := (-1, 1)$. We define the layer Ω of width $2a$ built over the surface Σ as the image of

$$\mathcal{L} : \mathbb{R}^2 \times I \rightarrow \mathbb{R}^3 : \{(s, y, u) \mapsto \mathcal{L}_0(s, y) + aun(s)\}.$$

We will always assume that

$$a < \varrho_m := \|\kappa\|_\infty^{-1} \quad \text{and } \Omega \text{ does not intersect itself.} \quad \langle A1 \rangle$$

Under these conditions, \mathcal{L} is a diffeomorphism onto Ω as one can see, e.g., from the formula for the metric G on Ω , induced by the immersion \mathcal{L} , that reads

$$(G_{ij}) = \begin{pmatrix} (G_{\mu\nu}) & 0 \\ 0 & a^2 \end{pmatrix}, \quad (G_{\mu\nu}) = \begin{pmatrix} f_a(s, u)^2 & 0 \\ 0 & 1 \end{pmatrix}; \quad i, j \in \{s, y, u\},$$

where $f_a(s, u) := 1 - auk(s)$. The assumption $\langle A1 \rangle$ implies, in particular, that $f_a(s, u) > 1 - a\|\kappa\|_\infty > 0$ holds for all $(s, u) \in \mathbb{R} \times I$.

Remark 2.1. Note that one can use $v = au \in (-a, a)$ as a natural transverse variable. The choice we made is suitable in situations when we want to discuss asymptotic properties of thin layers.

2.2 Dirichlet magnetic Laplacian

The main object of our interest is the magnetic Laplacian on Ω subject to the Dirichlet boundary condition,

$$-\Delta_{D,A}^\Omega = (-i\nabla + A)^2 \text{ (in the form sense)}, \quad Q(-\Delta_{D,A}^\Omega) = \mathcal{H}_{A,0}^1(\Omega, dx dy dz),$$

with a special choice of the vector potential, $A = B_0(0, x, 0)$, $B_0 > 0$, that corresponds to the homogeneous magnetic field $B = (0, 0, B_0)$. Using the unitary transform $\tilde{U} : L^2(\Omega, dx dy dz) \rightarrow L^2(\mathbb{R}^2 \times I, d\Omega)$, $\psi \mapsto \psi \circ \mathcal{L}$, we may identify $-\Delta_{D,A}^\Omega$ with the self-adjoint operator \hat{H} defined, in the form sense, on $L^2(\mathbb{R}^2 \times I, d\Omega)$ by

$$\begin{aligned} \hat{H} = & -f_a(s, u)^{-1} \partial_s f_a(s, u)^{-1} \partial_s + (-i\partial_y + \tilde{A}_2(s, u))^2 \\ & - a^{-2} f_a(s, u)^{-1} \partial_u f_a(s, u) \partial_u, \end{aligned}$$

where $\tilde{A} = (D\mathcal{L})^T A \circ \mathcal{L} = (0, \tilde{A}_2, 0)$ with

$$\tilde{A}_2(s, u) = B_0(x(s) - au\dot{z}(s)).$$

By another unitary transform, $U : L^2(\mathbb{R}^2 \times I, d\Omega) \rightarrow L^2(\mathbb{R}^2 \times I, d\Sigma du)$, $\psi \mapsto a^{1/2} f_a^{1/2} \psi$, we pass to the unitarily equivalent operator defined, again in the form sense, as

$$\tilde{H} = U \hat{H} U^{-1} = -\partial_s f_a(s, u)^{-2} \partial_s + (-i \partial_y + \tilde{A}_2(s, u))^2 - a^{-2} \partial_u^2 + V(s, u),$$

where

$$V(s, u) = -\frac{1}{4} \frac{\kappa(s)^2}{f_a(s, u)^2} - \frac{1}{2} \frac{au\ddot{\kappa}(s)}{f_a(s, u)^3} - \frac{5}{4} \frac{a^2 u^2 \dot{\kappa}(s)^2}{f_a(s, u)^4}.$$

These formulæ are easy to derive, see e.g. [9, 20] or [11, Sec. 1.1]. Remark that we needed C^4 -smoothness of Σ to write V down as an operator. Nevertheless, \tilde{H} may be introduced via its quadratic form for any C^3 surface.

The translational invariance of Ω makes it possible to pass finally to still another unitarily equivalent form of the operator by means of the Fourier–Plancherel transform in the y variable,

$$(\mathcal{F}_{y \rightarrow \xi} \psi)(s, \xi, u) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-iy\xi} \psi(s, y, u) dy,$$

which yields

$$H := \mathcal{F}_{y \rightarrow \xi} \tilde{H} \mathcal{F}_{y \rightarrow \xi}^{-1} = -\partial_s f_a(s, u)^{-2} \partial_s + (\xi + \tilde{A}_2(s, u))^2 - a^{-2} \partial_u^2 + V(s, u).$$

For a fixed $\xi \in \mathbb{R}$, we define

$$H[\xi] := -\partial_s f_a(s, u)^{-2} \partial_s + (\xi + \tilde{A}_2(s, u))^2 - a^{-2} \partial_u^2 + V(s, u), \quad (2.2)$$

which allows us to write our Hamiltonian in the form of a direct integral,

$$H = \int_{\mathbb{R}}^{\oplus} H[\xi] d\xi, \quad (2.3)$$

where ξ is the momentum of the motion in the y direction. Note that since $d\Sigma = ds \wedge dy$, the operator H and its fiber $H[\xi]$ act in $L^2(\mathbb{R}^2 \times I, ds d\xi du)$ and $L^2(\mathbb{R} \times I, ds du)$, respectively.

3 Absolute continuity of the magnetic Laplacian

As mentioned in the introduction we are interested in situations when the confinement causes a magnetic transport manifested through the absolute continuity of the spectrum. Our aim is to describe several classes of layers Ω for which the spectrum $-\Delta_{D,A}^\Omega$ is purely absolutely continuous. Since this operator is unitarily equivalent to the above described H which in turn decomposes into a direct integral with fibers $H[\xi]$, by [24, Thm XIII.86], it is sufficient to prove that

- (a) the family $\{H[\xi] \mid \xi \in \mathbb{R}\}$ is analytic with respect to ξ in the sense of Kato,
- (b) the resolvent of $H[\xi]$ is compact for any $\xi \in \mathbb{R}$,
- (c) no eigenvalue branch of $H[\xi]$ is constant in the variable ξ .

In the following subsections we address consecutively each of these points.

Remark 3.1. In fact, to see that (a)–(c) are sufficient conditions for the absolute continuity of H , we have to modify [24, Thm XIII.86] slightly. In particular, the variable ξ in the direct integral (2.3) runs over a non-compact interval and the eigenvalues branches are not necessarily bounded. An alternative short proof is based on the main result of [12] which says that under (a) and (b), $\sigma_{sc}(H) = \emptyset$ and $\sigma_p(H)$ may consist of isolated points without finite accumulation point only, and moreover, each one of these eigenvalues is of infinite multiplicity.

Since there are no finite accumulation points in $\sigma_p(H)$ and $\sigma_{sc}(H) = \emptyset$, we have $\sigma(H) = \sigma_p(H) \cup \sigma_{ac}(H)$, and consequently, it remains to check that $\sigma_p(H) = \emptyset$. By [24, Thm XIII.85], λ is an eigenvalue of H iff $\text{meas}_1\{\xi \in \mathbb{R} \mid \lambda \in \sigma_p(H[\xi])\} > 0$, where meas_1 stands for the one-dimensional Lebesgue measure. Since $H[\xi]$ has compact resolvent by (b) and it is unbounded but lower bounded, its spectrum consists of a sequence of eigenvalues of finite multiplicity which can accumulate only at $+\infty$. This together with (a) implies that there are countably many eigenvalue branches. Consequently, $\lambda \in \sigma_p(H)$ iff there is an eigenvalue branch, say $\lambda_p[\xi]$, such that $\text{meas}_1\{\xi \in \mathbb{R} \mid \lambda_p[\xi] = \lambda\} > 0$. By (a), the function $\lambda_p[\cdot] - \lambda$ is real analytic, and by (c), it is non-constant. This means that the equation $\lambda_p[\xi] = \lambda$ may have at most countable number of solutions and proves thus the claim.

3.1 Analyticity in ξ

For any $\xi_0 \in \mathbb{R}$, we have in the form sense

$$H[\xi] = H[\xi_0] + p_\xi,$$

where the quadratic form p_ξ is given by

$$p_\xi(\psi) = (\xi - \xi_0)^2 \|\psi\|^2 + 2(\xi - \xi_0) \langle \psi, (\xi_0 + \tilde{A}_2)\psi \rangle. \quad (3.4)$$

For any $\delta > 0$, one easily gets from here

$$\begin{aligned} |p_\xi(\psi)| &\leq (\xi - \xi_0)^2 (1 + \delta^{-1}) \|\psi\|^2 + \delta \|(\xi_0 + \tilde{A}_2)\psi\|^2 \\ &\leq (\xi - \xi_0)^2 (1 + \delta^{-1}) \|\psi\|^2 + \delta \langle \psi, H[\xi_0]\psi \rangle + \delta \langle \psi, V_- \psi \rangle, \end{aligned}$$

where V_- stands for the negative part of $V =: V_+ - V_-$. If we assume that

$$V_- \text{ is relatively form bounded by } H[0], \quad \langle A2 \rangle$$

which is equivalent to the assumption that V_- is relatively form bounded by $H[\xi_0]$, then $H[\xi]$ is lower bounded and p_ξ is infinitesimally form bounded by $H[\xi_0]$. In combination with (3.4) this implies that $H[\xi]$ forms an analytic family of type (B), in particular, that $H[\xi]$ is an analytic family in the sense of Kato [19].

3.2 Compactness of the resolvent

Assume now, in addition, that

$$\left| \lim_{s \rightarrow \pm\infty} x(s) \right| = +\infty, \quad V_- \in L^\infty(\mathbb{R} \times I, dsdu). \quad \langle A3 \rangle$$

Remark that under the second assumption, $\langle A2 \rangle$ is trivially satisfied. By [24, Thm XIII.64], the fiber $H[\xi]$ has compact resolvent if and only if the set

$$\mathcal{C}_{H[\xi],b} := \{\psi \in Q(H[\xi]) \mid \|\psi\| \leq 1, \langle \psi, H[\xi]\psi \rangle \leq b\}$$

is compact for all b . By $\langle A1 \rangle$ we have $f_a(s, u) \leq 1 + a\|\kappa\|_\infty =: d$, and therefore

$$H[\xi] \geq -d^{-2}\partial_s^2 + (\xi + \tilde{A}_2)^2 - a^{-2}\partial_u^2 - \|V_-\|_\infty.$$

Moreover, there exists clearly an s_0 such that for all $s : |s| \geq s_0$, we have

$$(\xi + \tilde{A}_2)^2 \geq \left(\xi + \frac{B_0}{2}x(s) \right)^2.$$

If we introduce the constant

$$K := \sup_{|s| < s_0, u \in I} \left| (\xi + \tilde{A}_2(s, u))^2 - \left(\xi + \frac{B_0}{2}x(s) \right)^2 \right|,$$

then we can estimate $H[\xi]$ from below as follows,

$$H[\xi] \geq -d^{-2}\partial_s^2 + \left(\xi + \frac{B_0}{2}x(s) \right)^2 - a^{-2}\partial_u^2 - K - \|V_-\|_\infty =: H_- - K - \|V_-\|_\infty.$$

The operator H_- on the right-hand side decomposes into a sum of the one-dimensional harmonic oscillator Hamiltonian and the Dirichlet Laplacian on I , hence it has compact resolvent [24, Thm XIII.64], and consequently, $\mathcal{C}_{H_-,b}$ is compact for any b . Clearly, $\mathcal{C}_{H[\xi],b} \subset \mathcal{C}_{H_-,b+K+\|V_-\|_\infty}$, which means that $\mathcal{C}_{H[\xi],b}$ has to be precompact, but at the same time $\mathcal{C}_{H[\xi],b}$ is closed, cf. the proof of [24, Thm XIII.64], and thus compact.

Remark 3.2. By [19, Thm VII.4.3], the fiber $H[\xi]$ has compact resolvent either for all ξ or for no ξ at all. It is easy, however, to demonstrate the resolvent compactness directly for any fixed ξ in the above described way.

3.3 Non-constancy of the eigenvalues

In the following section, we will demonstrate this property for several special but still rather wide classes of layers indicated in the introduction. Specifically, we will be concerned with the following cases:

- (i) *a one-sided-fold layer*: $\lim_{s \rightarrow \pm\infty} x(s) = +\infty$ or $\lim_{s \rightarrow \pm\infty} x(s) = -\infty$,
- (ii) *a bent, asymptotically flat layer*: $\dot{x}(s) = \alpha_+$ for all large enough positive s and $\dot{x}(s) = \alpha_-$ for all large enough negative s , where $\alpha_{\pm} \in (0, 1]$, $\alpha_+ \neq \alpha_-$,
- (iii) *a thin non-planar layer*: a is sufficiently small.

Let us remark that there are basically two methods how to demonstrate non-constancy of the eigenvalues. The first relies on the Feynman-Hellmann formula that gives the derivative of an eigenvalue with respect to a parameter. It is useful in situations when the curvature is compactly supported. The other method is based on some type of a comparison argument that should give asymptotic behavior of the eigenvalues at $\pm\infty$. It is usually applied in situations when the curvature behaves differently at $\pm\infty$.

4 Special classes of layers

Recall that throughout the paper we assume $\langle A0 \rangle$ and $\langle A1 \rangle$. In this section, we will suppose that $\langle A3 \rangle$ (which in turn implies $\langle A2 \rangle$) is always satisfied, too.

4.1 One-sided-fold layer

For the sake of definiteness, assume that $\lim_{s \rightarrow \pm\infty} x(s) = +\infty$. Then

$$\lim_{s \rightarrow \pm\infty} \tilde{A}_2(s, u) = +\infty$$

holds for all $u \in I$. Recall that $\dot{x}(s)^2 + \dot{z}(s)^2 = 1$ which makes it possible to estimate $\tilde{A}_2(s, u) \geq B_0(x(s) - a)$. This in turn implies that $(\tilde{A}_2)_-$ is compactly supported. For any $\xi > 0$ we have

$$(\xi + \tilde{A}_2)^2 \geq \xi^2 + \tilde{A}_2^2 - 2\xi\|(\tilde{A}_2)_-\|_{\infty},$$

and therefore

$$H[\xi] \geq -\partial_s f_a^{-2} \partial_s - a^{-2} \partial_u^2 + \tilde{A}_2^2 + \xi^2 - 2\xi\|(\tilde{A}_2)_-\|_{\infty} - \|V_-\|_{\infty}.$$

The first three terms on the right-hand side are positive and for the remaining part, independent of s and u , we have

$$\lim_{\xi \rightarrow +\infty} \left(\xi^2 - 2\xi \|(\tilde{A}_2)_-\|_\infty - \|V_-\|_\infty \right) = +\infty.$$

Thus to any $C > 0$ there is a $\xi_C \in \mathbb{R}$ such that $H[\xi] > C$ holds for all $\xi > \xi_C$, and consequently, no eigenvalue branch may be constant as a function of ξ .

4.2 A digression: flat layers

Before proceeding further we are going to show that a mere rotation of the layer around an axis perpendicular to the magnetic field is not sufficient – with one notable exception – to produce a transport in the considered system. The decisive quantity is the tilt angle between the field direction and the layer.

4.2.1 Inclined layer not parallel with the magnetic field

Let γ stands for the angle between the magnetic field B and the normal vector to the layer n . In the case of the unperturbed system, a planar layer with a perpendicular field ($\gamma = 0$), it is straightforward to see by separation of variables in the cylindrical coordinates that the spectrum of H is purely point. The same is true if $\dot{x}(s) = \cos \gamma$, $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus \{0\}$, holds for all $s \in \mathbb{R}$. Indeed, each of the fiber operators

$$H[\xi] = -\partial_s^2 + (\xi + B_0(s \cos \gamma - au \sin \gamma))^2 - a^{-2} \partial_u^2$$

is unitarily equivalent to $H[0]$, as one can verify employing a unitary transform $\psi(s, u) \mapsto \psi(s + \xi/(B_0 \cos \gamma), u)$. Using then [24, Thm XIII.85] in combination with the fact that $H[0]$ has compact resolvent, we conclude that $\sigma(H) = \sigma_p(H) = \sigma(H[0])$ consists of infinitely degenerate eigenvalues.

For $\gamma = 0$, this yields an alternative way to determine the spectral character in the unperturbed case by observing that

$$H[\xi] = -\partial_s^2 + (\xi + B_0 s)^2 - a^{-2} \partial_u^2$$

decomposes then into the sum of the Hamiltonian of the harmonic oscillator with the origin shifted by $-\xi/B_0$ and the Dirichlet Hamiltonian on the line segment I . For its eigenpairs, $(\lambda_{m,n}[\xi], \psi_{m,n}[\xi])$, we have

$$\begin{aligned} \lambda_{m,n}[\xi] &\equiv \lambda_{m,n} = B_0(2m+1) + \left(\frac{n\pi}{2a}\right)^2 \\ \psi_{m,n}[\xi](s, u) &= \psi_m(s + \xi/B_0) \chi_n(u), \end{aligned}$$

with

$$\psi_m(x) = (2^m m!)^{-1/2} (B_0/\pi)^{1/4} e^{-B_0 x^2/2} H_m(B_0^{1/2} x), \quad (4.5)$$

where H_m stands for the m th Hermite polynomial, and

$$\chi_n(u) = \begin{cases} \cos(n\pi u/2) & \text{if } n \text{ is odd} \\ \sin(n\pi u/2) & \text{if } n \text{ is even.} \end{cases}$$

Here $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and all the eigenfunctions are normalized to one in the respective Hilbert spaces.

Since $\lambda_{m,n}[\xi]$ is independent of ξ , all the eigenvalues of H have infinite multiplicity. Let us ask about additional degeneracies, that is, about the multiplicity of eigenvalues of $H[\xi]$. Assume that $\lambda_{m,n} = \lambda_{\tilde{m},\tilde{n}}$ for some $m, \tilde{m} \in \mathbb{N}_0$ and $n, \tilde{n} \in \mathbb{N}$ such that $m \neq \tilde{m}$ and $n \neq \tilde{n}$. This means that

$$\theta m + n^2 = \theta \tilde{m} + \tilde{n}^2,$$

where $\theta := 8B_0(a/\pi)^2$, which implies that θ is a positive rational, $\theta \in \mathbb{Q}_+$. Conversely, if this is the case then $\theta = p/q$, for some $p, q \in \mathbb{N}$, and the equation

$$\frac{p}{q}(m - \tilde{m}) = (\tilde{n} - n)(n + \tilde{n}) \quad (4.6)$$

has an infinite number of solutions $\{m, \tilde{m}, n, \tilde{n}\} \in \mathbb{N}_0^2 \times \mathbb{N}^2$ satisfying

$$\tilde{n} - n = p, \quad m - \tilde{m} = q(n + \tilde{n}),$$

in other words, every eigenvalue of $H[\xi]$ has infinite multiplicity.

On the other hand, in the case $\theta \in \mathbb{R}_+ \setminus \mathbb{Q}$ the spectrum $H[\xi]$ is simple but it becomes ‘denser’ as the energy increases. In other words, the eigenvalue gaps have no positive lower bound. Indeed, by Dirichlet’s approximation theorem, for all $N \in \mathbb{N}$ there exist $p, q \in \mathbb{N}$ such that $q \leq N$ and $|q\theta - p| \leq (N+1)^{-1}$. We will be concerned about large values of N , so we may assume that $p \geq 3$. The equation (4.6) has a infinite number of solutions $\{m, \tilde{m}, n, \tilde{n}\} \in \mathbb{N}_0^2 \times \mathbb{N}^2$ satisfying

$$\tilde{n} + n = p, \quad 0 < |\tilde{n} - n| \leq 2, \quad m - \tilde{m} = q(\tilde{n} - n).$$

For any of these solutions we obtain

$$\begin{aligned} |\theta m + n^2 - (\theta \tilde{m} + \tilde{n}^2)| &= \left| \left(\frac{p}{q} + \theta - \frac{p}{q} \right) (m - \tilde{m}) + n^2 - \tilde{n}^2 \right| \\ &= \left| \theta - \frac{p}{q} \right| |m - \tilde{m}| \leq \frac{1}{q(N+1)} 2q = \frac{2}{N+1}. \end{aligned}$$

We infer that for any $\varepsilon > 0$ there are infinitely many pairs of eigenvalues $\lambda_{m,n}$, $\lambda_{\tilde{m},\tilde{n}}$ with the property that

$$|\lambda_{m,n} - \lambda_{\tilde{m},\tilde{n}}| < \varepsilon$$

which proves our claim.

4.2.2 Inclined layer parallel with the magnetic field

The situation changes when the tilted layer has the right angle with the original one becoming thus parallel to the field direction. Then we have $\gamma = \frac{\pi}{2}$, and therefore

$$H[\xi] = -\partial_s^2 + (\xi - B_0 a u)^2 - a^{-2} \partial_u^2 = \overline{T_1 \otimes I + I \otimes T_2[\xi]}, \quad (4.7)$$

where

$$T_1 := -\partial_s^2, \quad T_2[\xi] := -a^{-2} \partial_u^2 + (\xi - B_0 a u)^2,$$

and we decomposed $L^2(\mathbb{R} \times I, ds du) = L^2(\mathbb{R}, ds) \otimes L^2(I, du) =: \mathcal{H}_1 \otimes \mathcal{H}_2$. The other choice $\gamma = -\frac{\pi}{2}$ gives rise to a unitarily equivalent operator. Since $\sigma(T_1) = \sigma_{ac}(T_1) = [0, +\infty)$ and $T_2[\xi]$ has a positive simple pure point spectrum, $\sigma(H[\xi]) = [\inf \sigma(T_2[\xi]), +\infty)$ for all $\xi \in \mathbb{R}$. In fact, we are going to prove that every $H[\xi]$ is purely absolutely continuous.

Let E_1 , E_2 , and E be spectral families of the operators T_1 , $T_2[\xi]$ and $H[\xi]$, respectively. Then for all decomposable vectors $f_1 \otimes f_2 \in \mathcal{H}_1 \otimes \mathcal{H}_2$ we have

$$\begin{aligned} \|E(t)(f_1 \otimes f_2)\|^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{t-s} ds \|E_1(u)f_1\|^2 \|E_2(s)f_2\|^2 \\ &= \int_{\mathbb{R}^2} \chi_{(-\infty, t]}(u+s) ds \|E_1(u)f_1\|^2 \|E_2(s)f_2\|^2 \\ &= \|E_1(\cdot)f_1\|^2 * \|E_2(\cdot)f_2\|^2((-\infty, t]); \end{aligned}$$

for the first equality we refer here to the proof of [26, Thm 8.34]. Using the Fubini theorem it is easy to check that the convolution of an absolutely continuous measure with another (Borel) measure is also absolutely continuous. Consequently, the absolutely continuous subspace of $\mathcal{H}_1 \otimes \mathcal{H}_2$ contains all finite linear combinations of the decomposable vectors. However, these functions form a dense subspace, and moreover, the absolutely continuous subspace is always closed. This allows us to infer that $H[\xi]$ is purely absolutely continuous.

Remark 4.1. The fact that the absolute continuity of one of the operators in the decomposition (4.7) implies the absolute continuity of the ‘full’ operator is a common folk knowledge usually referred to, not quite exactly, to standard sources, see e.g. [5]; we prefer to include the above simple proof.

Let $(\mu_n[\xi], \varphi_n[\xi])$, $n \in \mathbb{N}$, denote the eigenpairs of $T_2[\xi]$, where the eigenvalues are numbered in the ascending order. Some important properties of the $\mu_n[\xi]$ ’s are reviewed in [13]. In particular, $\mu_n[\xi]$ are even and strictly increasing for $\xi > 0$, thus they have the only stationary point at $\xi = 0$. Since $\sigma(H[\xi]) = [\mu_1[\xi], +\infty)$, $\sigma(H) = \sigma_{ac}(H) = [\mu_1[0], +\infty)$. Hence the bottom of $\sigma(H)$ is given by the first eigenvalue of the harmonic oscillator constrained to the line segment I . This

question was addressed repeatedly in the literature, see e.g. [6], and it is easy to see that the answer is given by the smallest solution of the equation

$${}_1F_1\left(-\frac{\mu}{4B_0} + \frac{1}{4}, \frac{1}{2}, B_0a^2\right) = 0 \quad (4.8)$$

with respect to μ . Here ${}_1F_1$ stands for the Kummer confluent hypergeometric function. If we denote this solution $\mu(B_0, a)$, then $\mu(B_0, a) = a^{-2}\mu(B_0a^2, 1)$, hence it is sufficient to inspect the dependence of μ on one of the parameters. The solution to the spectral condition (4.8) cannot be written in a closed form but can be found numerically, see Fig. 1. Moreover using known asymptotic properties of the Kummer functions [6] one can find the behavior of $\mu(B_0, 1)$,

$$\mu(B_0, 1) = \frac{\pi^2}{4} + \left(\frac{1}{3} - \frac{2}{\pi^2}\right) B_0^2 + \left(\frac{4}{45\pi^2} - \frac{20}{3\pi^4} + \frac{56}{\pi^6}\right) B_0^4 + \mathcal{O}(B_0^5) \quad (4.9)$$

as $B_0 \rightarrow 0$.

To derive the asymptotic behavior of $\mu(B_0, 1)$ at large values of B_0 we begin with the variational characterization of the lowest eigenvalue,

$$\mu(B_0, 1) = \mu_1[0]|_{a=1} = \inf_{\psi \in H_0^1(I), \|\psi\|_I=1} (\|\psi'\|_I^2 + B_0^2\|u\psi\|_I^2).$$

Let ψ_0 be the ground state of the one-dimensional harmonic oscillator, see (4.5) for its explicit normalized form, then clearly

$$\mu(B_0, 1) > \inf_{\psi \in H^1(\mathbb{R}), \|\psi\|_{\mathbb{R}}=1} (\|\psi'\|_{\mathbb{R}}^2 + B_0^2\|u\psi\|_{\mathbb{R}}^2) = \|\psi_0'\|_{\mathbb{R}}^2 + B_0^2\|u\psi_0\|_{\mathbb{R}}^2 = B_0. \quad (4.10)$$

If we put $\tilde{\psi}(u) := \psi_0(u) - \psi_0(1)$, $u \in I$, then $\tilde{\psi} \geq 0$, $\tilde{\psi}_0 \in H_0^1(I)$ and

$$\begin{aligned} \mu(B_0, 1) &\leq \frac{1}{\|\tilde{\psi}\|_I^2} \langle \tilde{\psi}, T_2[0]|_{a=1} \tilde{\psi} \rangle_I = \frac{1}{\|\tilde{\psi}\|_I^2} \langle \tilde{\psi}, B_0\tilde{\psi} + B_0\psi_0(1) - B_0^2\psi_0(1)u^2 \rangle_I \\ &\leq \frac{1}{\|\tilde{\psi}\|_I^2} \langle \tilde{\psi}, B_0\tilde{\psi} + B_0\psi_0(1) \rangle_I = B_0 + B_0\psi_0(1) \frac{1}{\|\tilde{\psi}\|_I^2} \langle \tilde{\psi}, 1 \rangle_I \\ &\leq B_0 + B_0\psi_0(1) \frac{1}{\|\tilde{\psi}\|_I} |I|^{1/2}. \end{aligned}$$

Next, for any fixed $\delta \in (0, 1)$ we have

$$\|\tilde{\psi}\|_I \geq \|\psi_0\|_I - \|\psi_0(1)\|_I = \|\psi_0\|_I - \psi_0(1)|I|^{1/2} \geq 1 - \delta$$

provided B_0 is sufficiently large. We conclude that, for any $\tilde{\delta} > 0$,

$$\mu(B_0, 1) \leq B_0 + (1 + \tilde{\delta})B_0\psi_0(1)|I|^{1/2} = B_0 + (1 + \tilde{\delta})\frac{\sqrt{2}}{\pi^{1/4}}B_0^{5/4}e^{-B_0/2}$$

as $B_0 \rightarrow +\infty$. Putting this together with (4.10) we arrive at the expansion

$$\mu(B_0, 1) = B_0 + \mathcal{O}(B_0^{5/4}e^{-B_0/2}) \quad (4.11)$$

valid as $B_0 \rightarrow +\infty$.

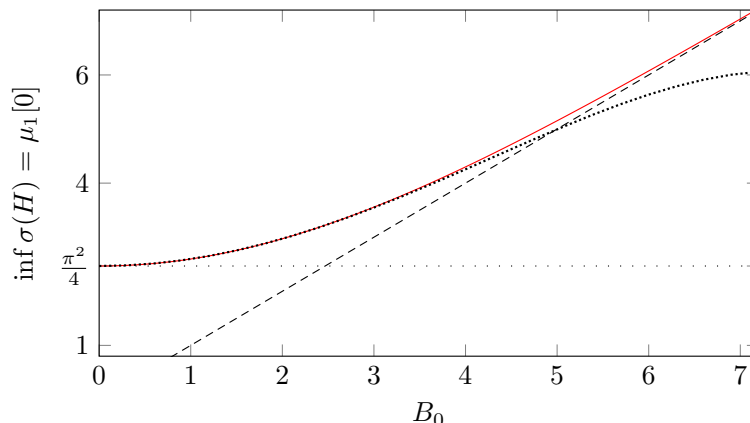


Figure 1: Bottom of $\sigma(H)$ as a function of B_0 (red solid line); $a = 1$. The black loosely dotted line corresponds to the first Dirichlet eigenvalue on $(-1, 1)$, i.e., $\frac{\pi^2}{4}$. The black densely dotted line corresponds to the asymptotic expansion (4.9) at $B_0 = 0$ and the black dashed line to the asymptotic expansion (4.11) at $B_0 = +\infty$.

In turn, H is also purely absolutely continuous by [24, Thm XIII.85]. The present situation is particular because H is also invariant with respect to s -translations, i.e. in the z direction. Using the partial Fourier–Plancherel transform in s , we find that \tilde{H} is unitarily equivalent to the operator having the following direct integral decomposition

$$\int_{\mathbb{R}}^{\oplus} T[\eta] d\eta := \int_{\mathbb{R}}^{\oplus} (\eta^2 + T[0]) d\eta, \text{ where } T[0] = (-i\partial_y - B_0 a u)^2 - a^{-2} \partial_u^2.$$

The physical contents of this decomposition is obvious: $T[0]$ is a purely absolutely continuous operator describing the edge-state-induced transport in a two-dimensional Dirichlet strip due to the perpendicular magnetic field [3, 13, 16], and so is $T[\eta]$. The difference between $T[\eta]$ and $T[0]$ which ‘adds’ to the absolute continuity of H is the square of the momentum in the z direction where the motion is free.

The decomposition (2.3) is also reflected in the unitary propagator for H which is given by a direct integral of the unitary propagators for $H[\xi]$, $\xi \in \mathbb{R}$. Since each

$H[\xi]$ is separated, we have

$$\exp(-itH[\xi]) = \overline{\exp(-itT_1) \otimes \exp(-itT_2[\xi])}$$

by [26, Thm 8.35]. In the z direction the evolution governed by the propagator with the well-known kernel

$$(\exp(-itT_1))(s, s') = (4\pi it)^{-1/2} \exp \frac{(s - s')^2}{4it},$$

while the second propagator in the tensor product decomposition can be expressed in terms of the eigenpairs of $T_2[\xi]$ describing the edge states as

$$\exp(-itT_2[\xi]) = \sum_{n \in \mathbb{N}} e^{-it\mu_n[\xi]} |\varphi_n[\xi]\rangle \langle \varphi_n[\xi]|.$$

The latter describes the well-known edge-current dynamics [16], the additional degree of freedom is a free motion of the wavepackets in the z direction having the usual properties, in particular, the spreading with time [2, Sec. 9.3].

4.3 Bent and asymptotically flat layers

Let us return now to geometrically nontrivial perturbations of the layer and assume they are localized at any fixed y cut, i.e., that the layer is flat for s outside a compact interval. Specifically, we suppose that $\dot{x}(s) = \alpha_+ > 0$ holds for all s large enough positive, $\dot{x}(s) = \alpha_- > 0$ for all s large enough negative, and $\lim_{s \rightarrow \pm\infty} \dot{z}(s) \geq 0$. Hence there are numbers s_0 and \tilde{s}_0 such that

$$\tilde{A}_2(s, u) = B_0 \left(\alpha_+(s - s_0) + x(s_0) - au\sqrt{1 - \alpha_+^2} \right).$$

holds for all $s \geq s_0$ and

$$\tilde{A}_2(s, u) = B_0 \left(\alpha_-(s - \tilde{s}_0) + x(\tilde{s}_0) - au\sqrt{1 - \alpha_-^2} \right). \quad (4.12)$$

holds similarly for all $s \leq \tilde{s}_0$. Recall that in view of the chosen layer parametrization we have $\alpha_{\pm} \leq 1$, their positivity means that we exclude the situation where the layer is asymptotically parallel with the magnetic field. With the future purpose in mind we also assume that *the magnitudes of the asymptotic slopes are different*, $\alpha_+ \neq \alpha_-$; without loss of generality we may suppose that $\alpha_- > \alpha_+ > 0$, since in the opposite case it is sufficient to change the layer parametrization replacing s by $-s$. Remark that we have chosen $\lim_{s \rightarrow \pm\infty} \dot{z}(s)$, that determine the signs of

the asymptotic slopes, to be non-negative just for definiteness. Changing the sign either of one or the both limits changes nothing in the considerations below.

Let us start with the first case assuming that the parameter s is sufficiently large and positive. Let $s_1 = s_1(\xi)$ be a solution of the equation

$$\xi + B_0(\alpha_+(s_1 - s_0) + x(s_0)) = 0.$$

Obviously, for all ξ sufficiently negative $s_1 = s_1(\xi)$ exists, being of course unique. In the following we restrict ourselves only to those values of ξ . It is straightforward to check that $\lim_{\xi \rightarrow -\infty} s_1(\xi) = +\infty$ and

$$\xi + \tilde{A}_2 = B_0 \left(\alpha_+(s - s_1) - au\sqrt{1 - \alpha_+^2} \right).$$

holds for all $s \geq s_0$. Consequently, the fiber $H[\xi]$ acts for all $s \geq s_0$ in the same way as the following positive operator,

$$H_{\alpha_+, s_1}(B_0) \equiv H_{\alpha_+, s_1} = -\partial_s^2 + B_0^2 \left(\alpha_+(s - s_1) - au\sqrt{1 - \alpha_+^2} \right)^2 - a^{-2}\partial_u^2.$$

Next we introduce the unitary transform $U_{s_1} : \psi(s, u) \mapsto \psi(s - s_1, u)$, then

$$U_{s_1}^{-1} H_{\alpha_+, s_1} U_{s_1} = H_{\alpha_+, 0} =: H_{\alpha_+} \equiv H_{\alpha_+}(B_0)$$

and $H_+[\xi] := U_{s_1}^{-1} H[\xi] U_{s_1}$ acts as

$$H_+[\xi] = \begin{cases} H_{\alpha_+} & s \geq s_0 - s_1 \\ -\partial_s f_a(s + s_1, u)^{-2} \partial_s + (\xi + \tilde{A}_2(s + s_1, u))^2 & s < s_0 - s_1 \\ -a^{-2} \partial_u^2 + V(s + s_1, u) & \end{cases}$$

Lemma 4.2. *Let ξ and $s_1 = s_1(\xi)$ be as above. Then, for all ξ sufficiently negative, there exist constants $C_{\pm}(\xi)$ and $K_{\pm} \in \mathbb{R}$, the latter being independent of ξ , such that $0 < C_-(\xi) < C_+(\xi)$,*

$$C_-(\xi)H_{\alpha_+} + K_- \leq H_+[\xi] \leq C_+(\xi)H_{\alpha_+} + K_+, \quad (4.13)$$

and the $C_{\pm}(\xi)$ have finite positive limits as $\xi \rightarrow -\infty$.

Proof. For all $s < s_0 - s_1$ we have

$$H_+[\xi] \leq -(1 - a\|\kappa\|_{\infty})^{-2} \partial_s^2 + (\xi + \tilde{A}_2(s + s_1, u))^2 - a^{-2} \partial_u^2 + \|V\|_{\infty}. \quad (4.14)$$

Since V is continuous and compactly supported, $\|V\|_{\infty} < +\infty$. Given $s \in \mathbb{R}$, put $f(s) := \xi + \tilde{A}_2(s + s_1, 0)$, then $f(0) = 0$, $f(s) = B_0 \alpha_+ s$ on $(s_0 - s_1, +\infty)$, $f(s) =$

$f(\tilde{s}_0 - s_1) + B_0 \alpha_- (s - \tilde{s}_0 + s_1)$ on $(-\infty, \tilde{s}_0 - s_1)$, and $\sup_{s, s' \in (\tilde{s}_0 - s_1, s_0 - s_1)} |f(s) - f(s')| = \sup_{s, s' \in (\tilde{s}_0, s_0)} |\tilde{A}_2(s, 0) - \tilde{A}_2(s', 0)| < +\infty$. Hence for all ξ large enough negative, i.e. for s_1 sufficiently large there exist $\tilde{C}_\pm = \tilde{C}_\pm(\xi)$ such that $0 < \tilde{C}_- < \tilde{C}_+$ and $\tilde{C}_+ s \leq f(s) \leq \tilde{C}_- s$ on $(-\infty, 0)$. Moreover, since $\sup_{s \in \mathbb{R}, u \in I} |\xi + \tilde{A}_2(s + s_1, u) - f(s)| = \sup_{s \in \mathbb{R}, u \in I} |\tilde{A}_2(s, u) - \tilde{A}_2(s, 0)| < +\infty$, we conclude that for all ξ sufficiently large negative there are $\hat{C}_\pm = \hat{C}_\pm(\xi)$ such that $0 < \hat{C}_- < 1 < \hat{C}_+$, and

$$\begin{aligned} \hat{C}_+ B_0 (\alpha_+ s - au \sqrt{1 - \alpha_+^2}) &\leq \xi + \tilde{A}_2(s + s_1, u) \\ &\leq \hat{C}_- B_0 (\alpha_+ s - au \sqrt{1 - \alpha_+^2}) < 0 \end{aligned}$$

holds on $(-\infty, s_0 - s_1) \times I$. A closer inspection shows that \hat{C}_\pm may be chosen in such a way that

$$\hat{C}_-(\xi) \nearrow \frac{\min\{\alpha_+, \alpha_-\}}{\alpha_+} = 1, \quad \hat{C}_+(\xi) \searrow \frac{\max\{\alpha_+, \alpha_-\}}{\alpha_+} = \frac{\alpha_-}{\alpha_+}$$

holds as $\xi \rightarrow -\infty$. Now we can proceed with estimate (4.14),

$$\begin{aligned} H_+[\xi] &\leq -(1 - a\|\kappa\|_\infty)^{-2} \partial_s^2 + \hat{C}_+^2 B_0^2 (\alpha_+ s - au \sqrt{1 - \alpha_+^2})^2 - a^{-2} \partial_u^2 + \|V\|_\infty \\ &\leq \max \left\{ (1 - a\|\kappa\|_\infty)^{-2}, \hat{C}_+^2 \right\} H_{\alpha_+} + \|V\|_\infty, \end{aligned}$$

for all $s < s_0 - s_1$. For $s \geq s_0 - s_1$, this bound holds trivially, too. In a similar manner one can estimate $H_+[\xi]$ from below putting

$$C_-(\xi) = \min \left\{ (1 + a\|\kappa\|_\infty)^{-2}, \hat{C}_-(\xi)^2 \right\}, \quad K_- = -\|V\|_\infty. \quad \square$$

If ξ is sufficiently large positive the argument is a simple modification of the above one. We define $\tilde{s}_1 = \tilde{s}_1(\xi)$ as the unique solution of the equation

$$\xi + B_0(\alpha_-(\tilde{s}_1 - \tilde{s}_0) + x(\tilde{s}_0)) = 0,$$

where \tilde{s}_0 was introduced in (4.12). Clearly, $\lim_{\xi \rightarrow +\infty} \tilde{s}_1(\xi) = -\infty$ and the operator $H_-[\xi] := U_{\tilde{s}_1}^{-1} H[\xi] U_{\tilde{s}_1}$ acts as

$$H_-[\xi] = \begin{cases} H_{\alpha_-} & s \leq \tilde{s}_0 - \tilde{s}_1 \\ -\partial_s f_a(s + \tilde{s}_1, u)^{-2} \partial_s + (\xi + \tilde{A}_2(s + \tilde{s}_1, u))^2 & s > \tilde{s}_0 - \tilde{s}_1. \\ -a^{-2} \partial_u^2 + V(s + \tilde{s}_1, u) \end{cases}$$

The operator pair $H_-[\xi]$ and H_{α_-} satisfies an estimate analogous to (4.13).

Now we are in position to prove an important convergence result.

Proposition 4.3. *Let $\mu > \|V_-\|_\infty$, then we have*

$$\lim_{\xi \rightarrow \mp\infty} \|(H_\pm[\xi] + \mu)^{-1} - (H_{\alpha_\pm} + \mu)^{-1}\| = 0.$$

Proof. It is sufficient to apply [25, Thm 2.3]. To make the paper self-contained we reproduce this result here: let $\{A[\alpha] \mid \alpha \in (-\infty, +\infty]\}$ be a one parametric family of lower-bounded selfadjoint operators on $L^2(M)$, where $M \subset \mathbb{R}^n$ is open, with the following properties

- (i) $C_0^\infty(M)$ is a core of $A[\alpha]$ for all $\alpha \in (-\infty, +\infty]$.
- (ii) There exist $C > 0$ and $K, \alpha_0 \in \mathbb{R}$ such that, for all $\alpha \geq \alpha_0$, $CA[+\infty] + K \leq A[\alpha]$.
- (iii) For any compact set $\mathcal{K} \subset M$, there exists $\alpha_{\mathcal{K}}$ such that, for all $\alpha \geq \alpha_{\mathcal{K}}$, $A[\alpha]|_{C_0^\infty(\mathcal{K})} = A[+\infty]|_{C_0^\infty(\mathcal{K})}$.
- (iv) $A[+\infty]$ has compact resolvent.

Then, for any $z \in \text{Res}(A[+\infty])$ and $\varepsilon > 0$, there exists $\alpha_{z,\varepsilon}$ such that for all $\alpha > \alpha_{z,\varepsilon}$, $z \in \text{Res}(A[\alpha])$ and

$$\|(A[\alpha] - z)^{-1} - (A[+\infty] - z)^{-1}\| < \varepsilon.$$

To deal with the limit $\xi \rightarrow -\infty$ we put $\alpha = -\xi$, $M = \mathbb{R} \times I$, $A[\alpha] = H_+[-\xi]$, and $A[+\infty] = H_{\alpha_+}$. The properties (i) and (ii) above are direct consequences of Lemma 4.2, (iii) is obvious from the definition of $H_+[\xi]$, and (iv) was proved in Sec. 3.2. The limit $\xi \rightarrow +\infty$ is treated in a similar manner. \square

Let us denote the eigenvalues of H_{α_+} , arranged in the ascending order with the multiplicity taken into account, by $\sigma_n(\alpha_+) \equiv \sigma_n(\alpha_+, B_0)$. Since the norm-resolvent convergence implies the convergence of eigenvalues, we see that in any neighborhood of $\sigma_n(\alpha_+)$, there is exactly the same number of eigenvalues of $H[\xi]$ as is the multiplicity of $\sigma_n(\alpha_+)$ in the spectrum of H_{α_+} , provided ξ is chosen sufficiently large negative. Similarly, in any neighborhood of $\sigma_n(\alpha_-)$, there is exactly the same number of eigenvalues of $H[\xi]$ as is the multiplicity of $\sigma_n(\alpha_-)$ in the spectrum of H_{α_-} , provided ξ is positive and sufficiently large. Moreover, if we fix $E > 0$ then for all $\sigma_n(\alpha_+)$ less than E we may choose the said neighborhoods to be disjoint and to prove that in the remaining gaps there are no eigenvalues of $H[\xi]$ for all ξ sufficiently large negative. Again, a similar statement holds true for large positive values of ξ .

In general, it may occur that the eigenvalue branches of $H[\xi]$ cross. It cannot happen, however, that a non-constant eigenvalue branch crosses a constant branch.

In fact, if there is a constant eigenvalue branch then it has to be isolated from the rest of the spectrum, cf. Remark 3.1 above, hence it makes sense to denote it as $\lambda_m[\xi]$, since it is indeed the m th eigenvalue of $H[\xi]$, with the multiplicity taken into account, for some $m \in \mathbb{N}$ and all $\xi \in \mathbb{R}$. If it is independent of ξ we would have $\lambda_m[\xi] = \sigma_m(\alpha_+, B_0) = \sigma_m(\alpha_-, B_0)$; our aim now is to find a sufficient condition under which this cannot happen.

To this aim, let us denote

$$T_\alpha(s, u) := B_0^2 \left(\alpha s - a\sqrt{1 - \alpha^2} u \right)^2,$$

and fix an $\varepsilon > 0$. Then, since

$$|asu| \leq \frac{1}{2} (\varepsilon s^2 + \varepsilon^{-1} a^2 u^2),$$

we get the inequalities

$$\begin{aligned} \left(\alpha^2 - \alpha\sqrt{1 - \alpha^2} \varepsilon \right) s^2 - a^2 \varepsilon^{-1} \alpha \sqrt{1 - \alpha^2} &\leq B_0^{-2} T_\alpha(s, u) \\ &\leq \left(\alpha^2 + \alpha\sqrt{1 - \alpha^2} \varepsilon \right) s^2 + a^2 \left(1 - \alpha^2 + \varepsilon^{-1} \alpha \sqrt{1 - \alpha^2} \right). \end{aligned}$$

This allows us to infer that

$$\begin{aligned} H_{\alpha_+}(B_0) &\leq H_1(B_0 g(\alpha_+, \varepsilon)) + B_0^2 a^2 \left(1 - \alpha_+^2 + \varepsilon^{-1} \alpha_+ \sqrt{1 - \alpha_+^2} \right) \\ H_{\alpha_-}(B_0) &\geq H_1(B_0 g(\alpha_-, -\varepsilon)) - B_0^2 a^2 \varepsilon^{-1} \alpha_- \sqrt{1 - \alpha_-^2}, \end{aligned}$$

where $g(\alpha, \varepsilon) := \sqrt{\alpha^2 + \alpha\sqrt{1 - \alpha^2} \varepsilon}$.

With respect to the first terms on the right-hand sides of the above inequalities, recall that $\sigma_m(1, B) = \lambda_{\tilde{m}, \tilde{n}}(B) = B(2\tilde{m} + 1) + a^{-2} E_{\tilde{n}}$ for some $\tilde{m} \in \mathbb{N}_0$ and $\tilde{n} \in \mathbb{N}$, where

$$E_{\tilde{n}} := \left(\frac{\tilde{n}\pi}{2} \right)^2. \quad (4.15)$$

Here, $E_{\tilde{n}}$ is the \tilde{n} th ‘transverse’ Dirichlet eigenvalue for $a = 1$. Furthermore, note the monotonicity with respect to the field: if $B < \tilde{B}$, then

$$\sigma_m(1, \tilde{B}) - \sigma_m(1, B) \geq \tilde{B} - B \quad (4.16)$$

holds for all $m \in \mathbb{N}$. This follows from the fact that we have

$$\begin{aligned} &|\{\lambda_{\tilde{m}, n}(\tilde{B}) \mid \exists \tilde{m} \in \mathbb{N}_0 : \lambda_{\tilde{m}, n}(\tilde{B}) \leq E\}| \\ &\leq |\{\lambda_{\tilde{m}, n}(B) \mid \exists \tilde{m} \in \mathbb{N}_0 : \lambda_{\tilde{m}, n}(B) \leq E - (\tilde{B} - B)\}|, \end{aligned}$$

for all $E > 0$ and $n \in \mathbb{N}$, where $|\cdot|$ stands for the cardinality of a set. Now using the minimax principle we obtain

$$\begin{aligned}\sigma_m(\alpha_+, B_0) &\leq \sigma_m(1, B_0 g(\alpha_+, \varepsilon)) + B_0^2 a^2 \left(1 - \alpha_+^2 + \varepsilon^{-1} \alpha_+ \sqrt{1 - \alpha_+^2}\right) \\ \sigma_m(1, B_0 g(\alpha_-, -\varepsilon)) &\leq \sigma_m(\alpha_-, B_0) + B_0^2 a^2 \varepsilon^{-1} \alpha_- \sqrt{1 - \alpha_-^2}.\end{aligned}$$

Combining this with (4.16) we arrive at the following claim.

Lemma 4.4. *Let $\alpha_- > \alpha_+ > 0$ and*

$$\varepsilon_0 := \frac{\alpha_-^2 - \alpha_+^2}{\alpha_+ \sqrt{1 - \alpha_+^2} + \alpha_- \sqrt{1 - \alpha_-^2}}.$$

Then we have $g(\alpha_-, -\varepsilon) > g(\alpha_+, \varepsilon)$ for all positive $\varepsilon < \varepsilon_0$. If, in addition,

$$a < a_0(\varepsilon) := \frac{1}{\sqrt{B_0}} \sqrt{\frac{g(\alpha_-, -\varepsilon) - g(\alpha_+, \varepsilon)}{1 - \alpha_+^2 + \varepsilon^{-1} (\alpha_+ \sqrt{1 - \alpha_+^2} + \alpha_- \sqrt{1 - \alpha_-^2})}},$$

holds for some $\varepsilon < \varepsilon_0$, then $\sigma_m(\alpha_+, B_0) < \sigma_m(\alpha_-, B_0)$.

Remark 4.5. One cannot maximize the threshold a_0 with respect to $\varepsilon \in (0, \varepsilon_0)$ analytically. However, it is possible to find a closed-form estimate by maximizing a lower bound. First of all, note that $0 \leq \alpha_{\pm} \sqrt{1 - \alpha_{\pm}^2} \leq 1/2$ holds for all $\alpha_{\pm} \in (0, 1]$. Hence $\varepsilon_0 \geq \tilde{\varepsilon}_0 := \alpha_-^2 - \alpha_+^2$, where $\tilde{\varepsilon}_0 < 1$, and

$$\begin{aligned}\sqrt{B_0} a_0(\varepsilon) &\geq \sqrt{\frac{\sqrt{\alpha_-^2 - \varepsilon/2} - \sqrt{\alpha_+^2 + \varepsilon/2}}{2\varepsilon^{-1}}} \\ &= \frac{1}{\sqrt{2}} \sqrt{\frac{\varepsilon(\alpha_-^2 - \alpha_+^2 - \varepsilon)}{\sqrt{\alpha_-^2 - \varepsilon/2} + \sqrt{\alpha_+^2 + \varepsilon/2}}} \geq \frac{1}{\sqrt{2}} \sqrt{\frac{\varepsilon(\alpha_-^2 - \alpha_+^2 - \varepsilon)}{\alpha_- + \sqrt{\alpha_+^2 + 1/2}}},\end{aligned}$$

for all $\varepsilon < \tilde{\varepsilon}_0$. The bound is maximal for $\varepsilon = (\alpha_-^2 - \alpha_+^2)/2 < \tilde{\varepsilon}_0$, so we arrive at

$$\sup_{\varepsilon \in (0, \varepsilon_0)} a_0(\varepsilon) \geq \frac{\alpha_-^2 - \alpha_+^2}{2\sqrt{2B_0} \sqrt{\alpha_- + \sqrt{\alpha_+^2 + 1/2}}} \geq \frac{\alpha_-^2 - \alpha_+^2}{2\sqrt{2} \sqrt{1 + \sqrt{3/2}} \sqrt{B_0}}.$$

By a *reductio ad absurdum* we can thus make the following conclusion.

Proposition 4.6. *Under the assumptions of Lemma 4.4, there are no constant eigenvalue branches of $H[\xi]$.*

4.4 Thin layers

The result of the previous section involved already a restriction to the layer thickness possibly going beyond the assumption $\langle A1 \rangle$, the severity of which depended on how much the layer was ‘broken’. Now we will go further and look what sufficient condition can be derived if the layer is very thin.

To begin with, recall that it was proved in [20] that if, in addition to $\langle A0 \rangle$ and $\langle A1 \rangle$,

$$\dot{\kappa}, \ddot{\kappa} \in L^\infty \tag{A4}$$

then for any k large enough,

$$\|(\tilde{H} - a^{-2}E_1 + k)^{-1} - (\tilde{h}_{\text{eff}} + k)^{-1} \oplus 0\| = \mathcal{O}(a)$$

holds as $a \rightarrow 0+$, with

$$\tilde{h}_{\text{eff}} = -\partial_s^2 + (-i\partial_y + B_0x(s))^2 - \frac{1}{4}\kappa^2(s)$$

acting on $L^2(\mathbb{R}^2, \text{dsd}y)$. Recall that E_1 is given by (4.15). Also remark that $\langle A0 \rangle$ combined with $\langle A4 \rangle$ yields $V \in L^\infty$, i.e., the second part of $\langle A3 \rangle$. Since we assume that the curvature κ is bounded, \tilde{h}_{eff} is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ and self-adjoint on

$$\text{Dom } \tilde{h}_{\text{eff}} = \{f \in L^2(\mathbb{R}^2, \text{dsd}y) \mid -\partial_s^2 f + (-i\partial_y + B_0x(s))^2 f \in L^2(\mathbb{R}^2, \text{dsd}y)\},$$

see [21].

Remark 4.7 (Magnetic field). The vector potential in the operator \tilde{h}_{eff} is $\tilde{A}_{\text{eff}} = (0, B_0x(s))$, and consequently,

$$\tilde{B}_{\text{eff}} = \text{curl} \tilde{A}_{\text{eff}} = B_0 \dot{x}(s) = B \cdot n.$$

Using the partial Fourier–Plancherel transform in the y variable, we turn the operator \tilde{h}_{eff} into

$$h_{\text{eff}} := -\partial_s^2 + (\xi + B_0x(s))^2 - \frac{1}{4}\kappa^2(s).$$

It is self-adjoint on its definition domain,

$$\text{Dom } h_{\text{eff}} = \{f \in L^2(\mathbb{R}^2, \text{dsd}\xi) \mid -\partial_s^2 f + (\xi + B_0x(s))^2 f \in L^2(\mathbb{R}^2, \text{dsd}\xi)\},$$

and decomposes into a direct integral,

$$h_{\text{eff}} = \int_{\mathbb{R}}^{\oplus} h_{\text{eff}}[\xi] \text{d}\xi,$$

where the fiber is given by

$$h_{\text{eff}}[\xi] = -\partial_s^2 + (\xi + B_0 x(s))^2 - \frac{1}{4}\kappa^2(s)$$

as an operator on $L^2(\mathbb{R}, ds)$. Using [20, Thm 6.3], we obtain further

$$\|(\tilde{H} - a^{-2}E_1 + k)^{-1} - (\tilde{H}_0 - a^{-2}E_1 + k)^{-1}\| = \mathcal{O}(a)$$

as $a \rightarrow 0+$, where \tilde{H}_0 is the ‘leading term’,

$$\tilde{H}_0 := \tilde{h}_{\text{eff}} \otimes I + I \otimes (-a^{-2}\partial_u^2),$$

and k has to be, of course, chosen large enough. In view of the unitarity of the Fourier–Plancherel transform, this implies

$$\|(H - a^{-2}E_1 + k)^{-1} - (H_0 - a^{-2}E_1 + k)^{-1}\| = \mathcal{O}(a),$$

where H_0 is defined similarly as \tilde{H}_0 but now with the help of h_{eff} . Since this operator also decomposes into a direct integral,

$$H_0 = \int_{\mathbb{R}}^{\oplus} (h_{\text{eff}}[\xi] \otimes I + I \otimes (-a^{-2}\partial_u^2)) \, d\xi =: \int_{\mathbb{R}}^{\oplus} H_0[\xi] \, d\xi,$$

we obtain the corresponding limiting relation for the fibers,

$$\|(H[\xi] - a^{-2}E_1 + k)^{-1} - (H_0[\xi] - a^{-2}E_1 + k)^{-1}\| = \mathcal{O}(a) \quad (4.17)$$

as $a \rightarrow 0+$. This follows from the fact that $\|\int_M^{\oplus} A[\xi]\| = \text{ess sup}_M \|A[\xi]\|$, cf. [24, Thm XIII.83], which also implies, in particular, that the error term on the right-hand side of (4.17) is uniform in $\xi \in \mathbb{R}$.

Assume that the operator $h_{\text{eff}}[\xi]$ has compact resolvent and all its eigenvalues are simple and analytic in ξ . This is fulfilled if, for instance, in addition to $\langle A0 \rangle$, which is sufficient for analyticity, we have $\dot{x}_{\pm} > 0$, cf. [25]. Here we employ for the sake of brevity the notation

$$\begin{aligned} \underline{f}_+ &:= \sup_{a \in \mathbb{R}} \text{ess inf}_{t \in (a, +\infty)} f(t) & \overline{f}_+ &:= \inf_{a \in \mathbb{R}} \text{ess sup}_{t \in (a, +\infty)} f(t) \\ \underline{f}_- &:= \sup_{a \in \mathbb{R}} \text{ess inf}_{t \in (-\infty, a)} f(t) & \overline{f}_- &:= \inf_{a \in \mathbb{R}} \text{ess sup}_{t \in (-\infty, a)} f(t). \end{aligned}$$

for a given $f \in L^\infty(\mathbb{R}; \mathbb{R})$. We denote the eigenvalues of $h_{\text{eff}}[\xi]$, arranged in the ascending order, as $\nu_m[\xi]$, $m \in \mathbb{N}$. Assume that they are non-constant as functions of ξ . A sufficient condition for this reads [25]

$$\dot{x}_{\pm} > 0 \wedge \dot{x}_+ \geq \overline{x}_- \wedge \left(\kappa_{-+}^2 - \overline{\kappa}_{-}^2 < 4B_0(\dot{x}_+ - \overline{x}_-) \right), \quad (4.18)$$

another one is obtained from (4.18) by changing the \pm indices to \mp everywhere. Our aim here is to derive another sufficient condition for the non-constancy which will be presented at the end of the section.

Under the stated assumptions on $h_{\text{eff}}[\xi]$, the spectrum of $H_0[\xi] - a^{-2}E_1$ consists of isolated eigenvalues

$$\gamma_{m,n}[\xi] = \nu_m[\xi] + a^{-2}(E_n - E_1).$$

By the minimax principle, $\nu_m[\xi] \geq -\frac{1}{4}\|\kappa^2\|_\infty$. We fix an energy value $E \in \mathbb{R}$ which we will refer to for brevity as threshold, then there exists an $a_E > 0$ such that

$$E < a^{-2}(E_2 - E_1) - \frac{1}{4}\|\kappa^2\|_\infty$$

holds for all $a < a_E$. Consequently, for these values of a , the spectrum of $H_0[\xi] - a^{-2}E_1$ strictly below E consists of the simple eigenvalues $\gamma_{m,1}[\xi] = \nu_m[\xi]$, $m = 1, 2, \dots, N[\xi]$, only. Note that $\max_{\xi \in \mathbb{R}} N[\xi] =: N_E < +\infty$.

There exist at least one compact interval I_E and a $\delta_E > 0$ such that $\nu_{N_E}[\xi] < E - 3\delta_E$ holds for all $\xi \in I_E$, because $\nu_{N_E}[\xi]$ is by assumption non-constant and analytic. For a fixed $m = 1, \dots, N_E$ we can then construct a tubular neighborhood $\mathcal{T}_m(\delta) := \{\nu_m[\xi] + t \mid \xi \in I_m, t \in (-\delta, \delta)\}$ with $\delta < \min\{\delta_E, \tilde{\delta}_E\}$, where

$$\tilde{\delta}_E := \frac{1}{4} \min_{m=1, \dots, N_E} \inf_{\xi \in I_E} \text{dist}(\nu_m[\xi], \sigma(H_0[\xi] - a^{-2}E_1))$$

is strictly positive. Furthermore, one can find an $\tilde{a}_E \in (0, a_E)$ such that for all $a < \tilde{a}_E$ there is exactly one eigenvalue branch of $H - a^{-2}E_1$ passing through each of the neighborhoods $\mathcal{T}_m(\delta)$, $m = 1, \dots, N_E$. Since $\nu_m[\xi]$ are non-constant, these eigenvalue branches must be non-constant as well, if we choose δ and consequently also \tilde{a}_E small enough.

Assume that there is a constant eigenvalue branch of $H - a^{-2}E_1$ below $E - \delta_E$. Then it must be, in particular, constant in the interval I_E , and thus it could not intersect with any of $\mathcal{T}_m(\delta)$, provided we chose δ and a as above. Moreover, by an easy perturbation theory consideration there are no eigenvalues of $H[\xi] - a^{-2}E_1$ in the remaining gaps whenever a is small enough. From this we can conclude that for any fixed threshold E , all the eigenvalue branches of H that lie (at least partially) below $E + a^{-2}E_1$ are non-constant provided the layer halfwidth a is sufficiently small.

Now we are going to derive the indicated new sufficient condition for non-constancy of the energy bands of the effective Hamiltonian. Let us assume that, in addition to $\langle A0 \rangle$,

$$x(s) = s \quad \text{for } s \leq 0; \quad \dot{x}(s) \geq 0 \quad \text{for } s > 0; \quad \underline{\dot{x}}_+ > 0; \quad x \neq \text{Id}. \quad (4.19)$$

It is convenient to write $x(s) = s + r(s)$, where $r(s) = \theta(s)r(s)$ with θ being the Heaviside step function. Clearly, $r(s) \leq 0$ and it is not identically zero. Note that this includes any perturbation of the planar layer that is compact in the x direction, since without loss of generality we may always suppose that such a perturbation is supported to the right of the origin. We have

$$\begin{aligned} h_{\text{eff}}[\xi] &= -\partial_s^2 + (\xi + B_0 s)^2 + 2B_0(\xi + B_0 s)r(s) + B_0^2 r^2(s) - \frac{1}{4}\kappa^2(s) \\ &= h_{\text{HO}}[\xi] + \theta(s) \left(B_0 r(s) (2\xi + B_0(s + x(s))) - \frac{1}{4}\kappa^2(s) \right). \end{aligned} \quad (4.20)$$

For any $\xi \geq 0$, $h_{\text{eff}}[\xi]$ is thus a non-positive perturbation of the shifted harmonic oscillator Hamiltonian $h_{\text{HO}}[\xi]$. Let us estimate the eigenvalue $\nu_m[0]$.

Let ψ_j be the j th eigenfunction of the harmonic oscillator Hamiltonian $h_{\text{HO}}[0]$ given by (4.5) and $S_m := \text{span}\{\psi_j \mid j = 1, \dots, m\}$; note that every function in S_m is real analytic on \mathbb{R} . Now, by the minimax principle,

$$\begin{aligned} \nu_m[0] &= \min_{\substack{S \subset \text{Dom}(h_{\text{eff}}[0]) \\ \dim S = m}} \max_{\substack{\psi \in S \\ \|\psi\|=1}} \langle \psi, h[0]\psi \rangle \leq \max_{\substack{\psi \in S_m \\ \|\psi\|=1}} \langle \psi, h[0]\psi \rangle \\ &\leq \max_{\substack{\psi \in S_m \\ \|\psi\|=1}} \langle \psi, h_{\text{HO}}[0]\psi \rangle + \max_{\substack{\psi \in S_m \\ \|\psi\|=1}} \langle \psi, \theta(s)(B_0^2 r(s)(s + x(s)) - \frac{1}{4}\kappa^2(s))\psi \rangle \\ &= B_0(2m + 1) + \max_{\substack{\psi \in S_m \\ \|\psi\|=1}} \int_0^{+\infty} (B_0^2 r(s)(s + x(s)) - \frac{1}{4}\kappa^2(s)) |\psi(s)|^2 ds. \end{aligned} \quad (4.21)$$

The last term on the right-hand side is negative, because the sub-integral function is non-positive everywhere and strictly negative on some interval, and the maximum of the integral is attained for some $\psi_{\text{max}} \in S_m$. Indeed, if the maximum was zero then ψ_{max} would be zero on the mentioned interval, and therefore due to the analyticity it would vanish on \mathbb{R} , which is a contradiction. We conclude that the sharp inequality $\nu_m[0] < B_0(2m + 1)$ holds for all $m \in \mathbb{N}$.

On the other hand, we have $\lim_{\xi \rightarrow +\infty} \nu_m[\xi] = B_0(2m + 1)$. To prove this claim we start with the unitary transform $U_\xi : \psi(s) \mapsto \psi(s - \xi/B_0)$ and introduce

$$\hat{h}_{\text{eff}}[\xi] = U_\xi h_{\text{eff}}[\xi] U_\xi^{-1} = \begin{cases} -\partial_s^2 + B_0^2 s^2 & s \leq \xi/B_0 \\ -\partial_s^2 + (\xi + B_0 x(s - \xi/B_0))^2 & \\ -\frac{1}{4}\kappa^2(s - \xi/B_0) & s > \xi/B_0 \end{cases}$$

and put $\hat{h}_{\text{eff}}[+\infty] := h_{\text{HO}}[0]$. Now we may apply the result of [25, Thm 2.3], which we have reproduced here as a part of the proof of Proposition 4.3, to the family

$\{\hat{h}_{\text{eff}}[\xi] \mid \xi \in (-\infty, +\infty)\}$. Let us focus on the assumption (ii) of the theorem. For all s sufficiently large we have $\dot{x}(s) > \frac{1}{2}\dot{x}_+$, and consequently, there is a $\xi_0 > 0$ such that for all $\xi > \xi_0$ and $s > 0$,

$$\xi + B_0 x(s - \xi/B_0) > \frac{1}{2} B_0 \dot{x}_+ s.$$

Using this estimate on the interval $(\xi/B_0, +\infty)$, we obtain

$$\frac{\dot{x}_+^2}{4} \hat{h}_{\text{HO}}[0] - \frac{1}{4} \|\kappa\|_\infty^2 \leq \hat{h}_{\text{eff}}[\xi].$$

The remaining assumptions are easy to verify. This makes it possible to infer from [25, Thm 2.3] that

$$\lim_{\xi \rightarrow +\infty} \|\hat{h}_{\text{eff}}[\xi]^{-1} - h_{\text{HO}}[0]^{-1}\| = 0,$$

which in turn implies that $\lim_{\xi \rightarrow +\infty} \nu_m[\xi]$ is just the m th eigenvalue of $h_{\text{HO}}[0]$. Our findings are summarized in the following claim.

Proposition 4.8. *Let the assumptions $\langle A0 \rangle$, $\langle A1 \rangle$, and $\langle A4 \rangle$ hold together with either (4.18) or (4.19). Then to any $E \in \mathbb{R}$ one can find an $a_E > 0$ such that no eigenvalue branch of the total Hamiltonian H that lies at least partially below $E + a^{-2}E_1$ can be constant as a function of ξ whenever $a < a_E$.*

5 An extension of the Iwatsuka model

While our main interest concerns magnetic transport in the Dirichlet layers, the considerations at the end of the previous section, in particular, the decomposition of the type (4.20) can be in combination with the minimax principle applied also to the classic Iwatsuka model. We start with the two-dimensional Hamiltonian

$$h_{\text{Iw}} = -\partial_x^2 + (-i\partial_y + A_y(x))^2 + W(x), \quad (5.22)$$

where

$$A_y(x) = \int_0^x B(t) dt.$$

Fix a $B_0 > 0$ and assume that $B(t) = B_0(1 + b(t))$ with

- (i) $b \in L_{\text{loc}}^2(\mathbb{R})$,
- (ii) $b(t) = 0$ for all $t < 0$,
- (iii) $\int_0^x b(t) dt \leq 0$ holds for all $x \geq 0$,

(iv) there are $\alpha \in (-1, 0)$, $x_1 \geq 0$ such that $\int_0^x b(t)dt > \alpha x$ holds for all $x \geq x_1$.

The potential $W \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ is such that $W(x) = \theta(x)W(x) \leq 0$. Under the stated integrability assumptions on b and W , the operator h_{Iw} is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, cf. [21].

Theorem 5.1. *Adopt the above assumptions together with $b \not\equiv 0 \vee W \not\equiv 0$, then h_{Iw} is purely absolutely continuous.*

Proof. As in the seminal paper of Iwatsuka [18] we start with a direct integral decomposition into fiber operators

$$h_{\text{Iw}}[\xi] = -\partial_x^2 + (\xi + A_y(x))^2 + W(x)$$

on $L^2(\mathbb{R}, dx)$. In the same manner as in [25] we show that $h_{\text{Iw}}[\xi]$ has compact resolvent and all its eigenvalues numbered in the ascending order as $\lambda_m[\xi]$, $m \in \mathbb{N}$, are simple and analytic on \mathbb{R} as functions of ξ . To prove the absolute continuity of h_{Iw} it suffices to demonstrate that no $\lambda_m[\cdot]$ is constant. We have

$$h_{\text{Iw}}[\xi] = h_{\text{HO}}[\xi] + \theta(x) \left(B_0 r(x) (2\xi + B_0(2x + r(x))) + W(x) \right), \quad (5.23)$$

where $h_{\text{HO}}[\xi]$ was introduced in (4.20) and

$$r(x) := \int_0^x b(t)dt.$$

Note that $r(x)$ is in view of (i) (absolutely) continuous. Using (iv), we find

$$2\xi + B_0(2x + r(x)) > 2\xi + B_0((2 + \alpha)x + R)$$

for all $x \geq 0$, where $R := \min_{x \in [0, x_1]} r(x) \leq 0$. Hence, if we set $\xi = -\frac{1}{2} B_0 R$ then $2\xi + B_0(2x + r(x)) = B_0(2x + r(x) - R) \geq 0$ holds for all $x \geq 0$. Taking the non-positivity of r and W into the account, we conclude that the second term in (5.23) is also non-positive. Moreover, the assumption $b \not\equiv 0 \vee W \not\equiv 0$ implies that it is strictly negative on some interval.

Mimicking the estimates in (4.21) with S_m being the span of first m eigenfunctions of $h_{\text{HO}}[-B_0 R/2]$, we arrive at $\lambda_m[-B_0 R/2] < B_0(2m + 1)$. Due to (iv), the second assumption of [25, Thm 2.3] is fulfilled which finally implies that $\lim_{\xi \rightarrow +\infty} \lambda_m[\xi] = B_0(2m + 1)$. \square

Let us recall that the family of magnetic fields considered above has a non-empty intersection with all the families studied earlier in the papers [18, 22, 10, 25] which, with the exception of the last one, treat the classic Iwatsuka model, $W \equiv 0$. Hence we obtain a nontrivial extension of the known results, with notably weak regularity assumptions comparing to the other sources. Note also the assumption (iii) crucial for the use of the minimax principle does not mean that the perturbation b of the constant magnetic field must be everywhere negative; it may be sign-changing and negative on a compact set only.

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